Functions
PSU Math Relays 2023

- There are a total of 22 questions on this test. For each problem place your answer in the appropriate blank on the answer sheet provided.
- All functions on this test are real-valued functions.
- Simplify each answer as far as possible.
- **No calculator** is allowed on this test.

For problems #1-3, use the piecewise-defined function

$$f(x) = \begin{cases} -2x - 3 & \text{if } x < 0, \\ 4 - x^2 & \text{if } x \geq 0. \end{cases}$$

1. Find the domain of $f(x)$.

2. Find the range of $f(x)$.

3. Find $f(f(f(f(2))))$.

For problems #4-6, use the parabolic function $h(x) = -x^2 + 6x - 7$.

4. Find the vertex of $h(x)$.

5. Find the axis of symmetry of $h(x)$.

6. Find the range of $h(x)$.

For problems #7-9, use the function $h(x) = \dfrac{3x^2 + 7}{x^2 + 3x - 4}$.

7. Find the domain of $h(x)$.

8. Find the vertical asymptotes of $h(x)$.

9. Find the horizontal asymptote of $h(x)$.

For problems #10-12, use the function $f(x) = \dfrac{2x + 1}{x - 3}$.

10. Find the inverse function $f^{-1}(x)$.

11. Find the composite function $(f \circ f)(x)$. Give the answer in lowest form.

12. Find the fixed points of $f(x)$, i.e., the values of x such that $f(x) = x$.

13. Let f and g be even functions and define $h(x) = f(g(x))$. Is h necessarily an even function?

14. Let f and g be odd functions and define $h(x) = f(g(x))$. Is h necessarily an odd function?

15. Let f be an even function and g an odd function. Define $h(x) = f(g(x))$. Is h necessarily an odd function?

16. Given a function $f(x)$, define $f(x)\big|_a^b = f(b) - f(a)$. Let $f(x) = \sqrt{20 - x^2}$. Find $f(x)\big|_2^4$.

17. Define $f(x) = -2x^2 + x - 3$. Simplify $\dfrac{f(x+h) - f(x)}{h}$.

18. Let $A(x)$ be the area of an equal lateral triangle inscribed in a circle of radius x. Write a formula for $A(x)$.

19. Let $P(n)$ be the number of primes in the interval $[1, n]$. For example, $P(5) = 3$ and $P(10) = 4$. Find $P(60)$.

20. A function f satisfies the relation $f(xy) = f(x) + f(y)$ for all values $x, y > 0$. Given $f(8) = 6$. Find $f(32)$.

21. A function G is defined recursively by

$$G(n) = nG(n - 1) - n^2 + 6$$

for all positive integers n. Let $G(1) = 0$. Find $G(5)$.

22. The function f is given by the table

x	1	2	3	4	5
$f(x)$	3	4	5	1	2

If $a_0 = 1$ and $a_{n+1} = f(a_n)$ for $n \geq 0$, find a_{2023}.