2023 Kansas Collegiate Math Competition

1. Suppose that x_1, x_2, x_3 are the roots of $x^3 + 2x + 2023 = 0$. Please prove or disprove: $x_1 + x_2 + x_3 = 0$.

Solution:
We know that the polynomial in question must factor as $(x - x_1)(x - x_2)(x - x_3)$. Expanding this, and equating the squared terms forces $x_1 + x_2 + x_3 = 0$.

2. Let x, y be chosen uniformly and independently from the interval $[0, 2]$. Let

$$A = \begin{bmatrix} x & x & 0 \\ x & x + x^2 + y & x^2 \\ 0 & x^2 & x^2 \end{bmatrix}$$

Please find the probability that $1 < \det(A) < 2$.

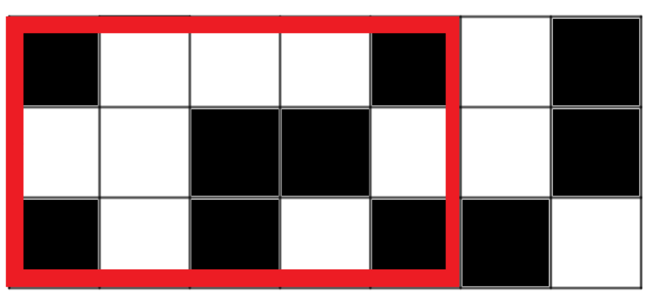
Solution:
The sample space for (x, y) is the square $[0, 2] \times [0, 2]$, which has area 4. Direct calculation shows that $\det(A) = x^3 y$. We need to find the area between the curves $y = \frac{1}{x^3}$ and $y = \frac{2}{x^3}$, or, equivalently, $x = y^{-\frac{1}{3}}$ and $x = (y/2)^{-\frac{1}{3}}$. We calculate

$$\int_{2^{-1/3}}^{1} 2 - x^{-3}\, dx + \int_{1}^{2} 2x^{-3} - x^{-3}\, dx = \frac{23}{8} - \frac{3 \cdot 2^{2/3}}{2}$$

Dividing by 4 yields the probability:

$$\frac{23}{32} - \frac{3 \cdot 2^{2/3}}{8} \approx .123 = 12.3\%.$$

3. Consider a 3-by-7 chessboard where the squares have been randomly colored black or white. Please prove that *any* such coloring must contain a rectangle (formed by the horizontal and vertical lines of the board such as the one outlined in the figure) whose four distinct unit corner squares are all of the same color.



Solution:
For the purposes of this problem, call a column *black* if it has more black squares than white squares, and *white* if it has more white squares than black squares.

Because there are 7 columns, there are either at least 4 black columns, or at least 4 white columns. Without loss of generality, suppose there are 4 black columns. Assume that each of these columns has a single white square. Now there are only 3 locations in each column for the white square - this leaves at least 2 columns with the white square in the same position. This means those columns are identically colored, thus giving the result. The case where one of the 4 columns is all black squares is covered by this case as well.

4. Let A, B, C, D denote 4 points in Euclidean space $\mathbb{R}^n$. Let AB denote the distance between A and B, and so on. Please prove that

$$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$$

and determine precisely when equality occurs.

Solution: Let a, b, c, d denote vectors from the origin to A, B, C, D, respectively. Recalling that the dot product of a vector with itself gives the square of its magnitude, the inequality in question reduces to

$$(a-c)^2 + (b-d)^2 + (a-d)^2 + (b-c)^2 \geq (a-b)^2 + (c-d)^2,$$

which is equivalent to the following true statement after some simplification

$$(a+b-c-d)^2 = (a+b-c-d) \cdot (a+b-c-d) \geq 0.$$

We will have equality precisely when $a+b-c-d=0$. That is, $a+b=c+d$, so that the 4 points form 4 vertices of a parallelogram.

5. Please determine all integer solutions to

$$a^2 + b^2 + c^2 = a^2 b^2.$$

Solution:
By reducing mod 4, it is easy to see that all of a, b, c must be even. Note that for any even integer n, $n^2 = 0 \mod 4$, whereas if n is odd, $n^2 = 1 \mod 4$. Therefore, if all of a, b, c are odd, the left hand side reduces to $3 \mod 4$, while the right hand side reduces to $1 \mod 4$. If exactly two of a, b, c are odd, the LHS reduces to 2, while the RHS reduces to either 1 or 0. If exactly one of a, b, c are odd, the LHS reduces to 1, while the RHS reduces to 0. So the only possible solutions can occur when a, b, c are all even.

Let $a = 2a_2$, $b = 2b_2$, $c = 2c_2$. Substituting, we get $4a_2^2 + 4b_2^2 + 4c_2^2 = 16a_2^2 b_2^2$, or

$$a_2^2 + b_2^2 + c_2^2 = 4a_2^2 b_2^2.$$

Now the RHS reduces to 0 mod 4. There are three terms on the LHS, each reducing to 0 or 1 mod 4. This forces them all to be 0 mod 4. Therefore, a_2, b_2, c_2

are all even. Writing $a_2 = 2a_3$, $b_2 = 2b_3$, $c_2 = 2c_3$ and substituting into the preceding equation yields

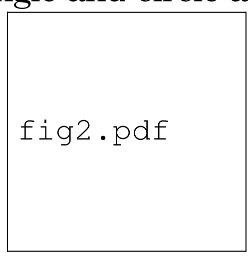
$$a_3^2 + b_3^2 + c_3^2 = 16a_3^2 b_3^2.$$

Again, we can conclude by reducing mod 4 that a_3, b_3, c_3 are all even. By repeating this process, we see that an arbitrarily large power of 2 divides each of a, b, c. This forces $a = b = c = 0$.

6. Prove that $e^\pi > \pi^e$.

Solution: Prove that the function $f(x) = \ln(x)/x$, $x \in (-\infty, \infty)$, has an absolute maximum at $x = e$. Then, use this fact to show that $f(\pi) < f(e)$, from which the desired inequality follows: $\ln(\pi)/\pi < 1/e$ implies $\pi < e^{\pi/e}$ and so $\pi^e < e^\pi$.

7. Consider an isosceles right triangle with perpendicular sides of fixed length a. Inscribe a rectangle and a circle inside the triangle as indicated in the figure below. Find the dimensions of the rectangle (and the radius of the circle) which make the total area of the rectangle and circle a maximum.



Solution: Denote the horizontal side of the rectangle by x and the radius of the circle by r. Then, due to the triangle being right and isosceles, the vertical side of the rectangle is equal to $(a - x)$. Moreover, drawing the rays from the center of the circle to the points of contact with the sides of the isosceles right triangle of side a, as well as the line from the center to the down left vertex of that triangle, we see that the area of the isosceles right triangle of side $(a - x)$ can be expressed as the sum of the areas of the four right triangles formed inside and also the rectangle of side r. Hence, as these four triangles are equal to each other, we have

$$4 \cdot \frac{1}{2} r(a - x - r) + r^2 = \frac{1}{2}(a - x)^2 \;\Rightarrow\; r^2 - 2(a - x)r + \frac{1}{2}(a - x)^2 = 0 \;\Rightarrow\; r = \frac{\left(2 \pm \sqrt{2}\right)(a - x)}{2}$$

and, since $r < a - x$, it follows that $r = \dfrac{\left(2 - \sqrt{2}\right)(a - x)}{2} = \dfrac{a - x}{2 + \sqrt{2}}$. Thus, we need to minimize the function

$$f(x) = \pi \frac{(a - x)^2}{(2 + \sqrt{2})^2} + x(a - x), \quad 0 \le x \le a.$$

Differentiating gives

$$f'(x) = 2\pi \frac{x-a}{(2+\sqrt{2})^2} + a - 2x = 2\Big(\underbrace{\frac{\pi}{(2+\sqrt{2})^2}}_{A} - 1\Big)(x-a) - a$$

thus $f'(x) = 0$ when $x - a = \dfrac{a}{2(A-1)}$ i.e. $x = \dfrac{(2A-1)a}{2(A-1)}$. Since

$$f(\frac{(2A-1)a}{2(A-1)}) = A(a-x)^2 + x(a-x) = \frac{Aa^2}{4(A-1)^2} - \frac{(2A-1)a}{2(A-1)} \cdot \frac{a}{2(A-1)} = \frac{a^2}{4(1-A)},$$

comparing this value with the endpoint values $f(a) = 0$ and $f(0) = Aa^2$ we conclude that the desired area is maximized at $x = \dfrac{(2A-1)a}{2(A-1)}$. This readily implies the corresponding value for the radius r.

8. Let the sequence $\{a_n\}$ be defined recursively by

$$a_1 = 0,\ a_2 = 1,\ a_n = (n-1)\left(a_{n-1} + a_{n-2}\right), n \geq 3.$$

Find an explicit formula for a_n and compute the limit $\lim\limits_{n\to\infty} \dfrac{a_n}{n!}$.

Solution: Rearranging the recursion relation, we have $a_n - na_{n-1} = -\left[a_{n-1} - (n-1)a_{n-2}\right]$. Proceeding this way, we eventually get $a_n - na_{n-1} = (-1)^{n-2}(a_2 - 2a_1) = (-1)^n$. Thus,

$$\frac{a_n}{n!} - \frac{a_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}$$

and so

$$\sum_{j=2}^{n} \left(\frac{a_j}{j!} - \frac{a_{j-1}}{(j-1)!}\right) = \sum_{j=2}^{n} \frac{(-1)^j}{j!} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

The sum on the left-hand side is telescoping and equal to $\dfrac{a_n}{n!}$. Thus,

$$a_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{a_n}{n!} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} = \frac{1}{e}.$$

9. Let A be a square matrix and suppose that there exists positive integers m and n such that $A^m = I$ and $A^n \neq I$. Calculate $\det\left(I + A + A^2 + \ldots + A^{m-1}\right)$.

Solution: Starting from the identity $\left(I + A + \ldots + A^{m-1}\right)(I - A) = I - A^m$ and using the hypothesis, we have

$$\left(I + A + \ldots + A^{m-1}\right)(I - A)\,\mathbf{x} = \mathbf{0}$$

for <u>any</u> vector $\mathbf{x}$. Then, since $A \neq I$ (otherwise we would have $A^j = I$ for all $j \in \mathbb{N}$ contradicting the hypothesis), there exists vector $\mathbf{x}$ such that $\mathbf{y} := (I - A)\mathbf{x} \neq \mathbf{0}$. But then $\mathbf{y}$ is an eigenvector of $(I + A + \ldots + A^{m-1})$ with eigenvalue zero, which means that $\det(I + A + \ldots + A^{m-1}) = 0$.

10. Consider the curve $y = \dfrac{1}{1+x^2}$, $x \geq 0$.

(i) Show that there exists a straight line intersecting the curve at $(0, 1)$ and tangent to the curve at some point with $x > 0$ but with no further intersections between the line and the curve.

(ii) By considering the area under the curve for $0 \leq x \leq 1$, show that $\pi > 3$.

(iii) By considering the volume formed by rotating the portion of the curve corresponding to $0 \leq x \leq 1$ about the y-axis, show also that $\ln 2 > \dfrac{2}{3}$.

Solution: (i) Since $y'(x) = -\frac{2x}{(1+x^2)^2}$, the tangent to the curve at (p, q) is $y = -2pq^2x + (q + 2pq^2p)$. In order for this tangent line to go through $(0, 1)$, we must have $1 = q + 2pq^2p$ which through the equation of the curve implies $p^4 = p^2$. Since $x \geq 0$, we deduce $p = 1$ and so the tangent line is $y = -\frac{1}{2}x + 1$. We can then check that the equation $-\frac{1}{2}x + 1 = \frac{1}{1+x^2}$ has no solutions other than $x = 0$ and $x = 1$ (double), and thus there is no further intersection between the tangent line and the curve.

(ii) Sketching the curve and its tangent found in part (i), we see that the area under the tangent and above $[0, 1]$ is less than the area under the curve and above $[0, 1]$. The first of these areas is equal to $3/4$, while the second one is calculated as

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}.$$

Thus, $3/4 < \pi/4$ which implies $\pi > 3$.

(iii) The volume formed by rotating the curve about the y-axis is given by

$$\int_0^1 2\pi x \cdot y dx = \pi \int_0^1 \frac{2x}{1+x^2} dx = \pi \ln 2.$$

This is greater than the volume formed by rotating the tangent line about the y-axis, which is equal to

$$\int_0^1 2\pi x \cdot y dx = \pi \int_0^1 x\,(-x + 2)\,dx = \frac{2\pi}{3}.$$

Thus, $2\pi/3 < \pi \ln 2$ implying that $\ln 2 > 2/3$.